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Interpolation by Uni- and Multivariate Generalized Splines

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Lagrange interpolation by finite-dimensional spaces of uni- and multivariate generalized spline functions (including polynomial splines) is studied. Using a condition of Schoenberg–Whitney type, it is shown how to change an almost interpolation set in order to obtain a set which admits unique Lagrange interpolation. Moreover, it is shown that every regular space of univariate generalized splines is a weak Chebyshev space if and only if every interpolation set can be characterized by a modified Schoenberg–Whitney type condition.

1. INTRODUCTION

We are interested in Lagrange interpolation using finite-dimensional spaces U of multivariate spline functions defined on a polyhedral region K in $\mathbb{R}^k$. The problem is to study configurations $T = \{t_1, ..., t_s\} \subset K$ where $s \leqslant n = \dim U$ and T is a set of distinct points such that

$$\dim U|_T = s.$$

In the case when $s = n$ this implies that for any given data $\{y_1, ..., y_n\}$ there exists a unique function $u \in U$ such that

$$u(t_i) = y_i, \qquad i = 1, ..., n.$$

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For univariate splines this problem is completely solved by Schoenberg and Whitney [4]. They have shown that interpolation is possible if and only if the points of interpolation are appropriately interlaced with the knots of the splines.

Recently, we have introduced a condition of Schoenberg–Whitney type and have shown in [7] that this condition characterizes all configurations T in K such that T is an almost interpolation set (Definition 2.1, Theorem 2.3). Applying this result we are now interested in the question of how to change an almost interpolation set in order to obtain a configuration which admits unique Lagrange interpolation (Theorem 3.1). In the case of bivariate linear splines we are able to develop a simple algorithm to determine interpolation sets (Theorem 3.4).

In Section 4 we consider the case of univariate generalized splines. We introduce the so-called strong condition of Schoenberg–Whitney type (SSW-condition) which is always necessary for any configuration T to be an interpolation set, and we are interested in which generalized spline spaces have the SSW-property; i.e. the SSW-condition is also sufficient for any T to be an interpolation set. In this context the class of weak Chebyshev subspaces of $C[a, b]$ plays an important role. Our main result states that a regular generalized spline space U has the SSW-property if and only if U is a weak Chebyshev space (Theorem 4.9).

Finally, in Section 5 we show that for a class of generalized splines our results stated in Section 4 extend results on interpolation by generalized splines and weak Chebyshev spaces given in [3] and by Davydov [2], respectively.

In the proof of Theorem 4.6 we present a method to construct a wide class of interpolation sets $T = \{t_1, ..., t_s\}$ for every $s \in \{1, ..., n\}$.

2. A Characterization of Almost Interpolation Sets

Recently, in [7] we have introduced a condition of Schoenberg–Whitney type to study the problem of interpolation by multivariate splines. To formulate it let

$$K := \bigcup_{i \in I} K_i$$

where K_i is a convex polyhedron in $\mathbb{R}^k$ $(k \geqslant 1)$, $i \in I$ and I denotes a finite subset of $\mathbb{N}$. Assume that $K_i \not\subset \bigcup_{j \in I \setminus \{i\}} K_j$. Moreover, assume that K is connected and for all $i, j \in I$, $i \neq j$ the intersection of K_i and K_j is empty or a common face (for more details see [7]).

Let any $p \in \mathbb{N} \cup \{0\}$ be given. For each $i = 1, ..., q$ suppose that V_i denotes a finite-dimensional linear subspace of $C^p(K_i)$ having the following property: Let $v \in V_i$ and $\tilde{t} \in Z(v) := \{t \in K_i : v(t) = 0\}$. If there exists $\varepsilon > 0$ such that $v(t) = 0$ for every $t \in K_i$ satisfying $\|t - \tilde{t}\| < \varepsilon$, then $v \equiv 0$ on K_i.

A prototype of V_i is the linear space Π_m of polynomials of total degree at most m ($m \in \mathbb{N}$) defined on $\mathbb{R}^k$.

We define the *linear space S of generalized splines of smoothness p* by

$$S := \{s \in C^p(K) \text{: for every } i \in I \text{ there exists } s_i \in V_i \text{ such that } s(t)|_{K_i} = s_i(t),\ t \in K_i\}. \tag{2.1}$$

Suppose that $\{u_1, ..., u_n\}$ denotes a system of linearly independent functions in S. Define

$$U := \text{span}\{u_1, ..., u_n\}.$$

In the following we consider subsets $T = \{t_1, ..., t_s\}$ of K ($s \leqslant n$) where we always assume that $t_i \neq t_j$ if $i \neq j$. We shall need the following notations.

DEFINITION 2.1. T is called an *interpolation set* (*I-set*) *with respect to U* (*w.r.t.U*) if

$$\dim U|_T = s.$$

Otherwise, T is called a *noninterpolation set* (*NI-set*) *w.r.t.U*.

T is called an *almost interpolation set* (*AI-set*) *w.r.t.U* if for any $\varepsilon > 0$ there exist points $\tilde{t}_i \in K$, $i = 1, ..., s$ satisfying $\|t_i - \tilde{t}_i\| < \varepsilon$, $i = 1, ..., s$ such that $\{\tilde{t}_1, ..., \tilde{t}_s\}$ is an I-set w.r.t.U.

Moreover, by card(M) we denote the cardinality of a finite subset M of K.

Let a subset $R := \bigcup_{j=1}^{l} K_{i_j}$ of K be given where $\{i_1, ..., i_l\} \subset I$. Then the *interior of R with respect to K* is defined by

$$\text{int}_K R := K \Big\backslash \bigcup_{i \in I \backslash \{i_1, ..., i_l\}} K_i.$$

In [7] we have introduced a property which has been inspired by a well-known condition of Schoenberg and Whitney ensuring uniqueness of Lagrange interpolation by univariate splines with fixed knots (Schoenberg and Whitney [4]; see also Schumaker [5]).

DEFINITION 2.2. Let $T = \{t_1, ..., t_s\} \subset K$ ($s \leqslant n$) and $U = \text{span}\{u_1, ..., u_n\}$. We say that T satisfies a *condition of Schoenberg–Whitney type* or T is an *SWT-set w.r.t.U* if

$$\text{card}(T \cap \text{int}_K R) \leqslant \dim U|_{\text{int}_K R}$$

for every choice of subsets $R := \bigcup_{j=1}^{l} K_{i_j}$ of K where $\{i_1, ..., i_l\} \subset I$.

The main result in [7] can now be stated as follows.

THEOREM 2.3 [7]. *Let* $T=\{t_1, ..., t_s\}$ *and* $U=\text{span}\{u_1, ..., u_n\}$. *Then* T *is an AI-set w.r.t.* U *if and only if* T *is an SWT-set w.r.t.* U.

We shall use this statement in Section 3 to treat the problem of how to change an AI-set to an I-set. To these investigations two results also given in [7] are very useful.

PROPOSITION 2.4 [7]. *Let* $T=\{t_1, ..., t_s\} \subset K$ $(s \leqslant n)$ *and assume that* U *denotes an* n*-dimensional subspace of* S. *Then the following conditions are equivalent.*

(i) T *is an SWT-set w.r.t.* U;

(ii) *For each basis* $\{u_1, ..., u_n\}$ *of* U *there is some permutation* σ *of* $\{1, ..., n\}$ *such that*

$$t_i \in S_{\sigma(i)} := \text{supp } u_{\sigma(i)} := \overline{\{t \in K: u_{\sigma(i)}(t) \neq 0\}}, \qquad i=1, ..., s.$$

PROPOSITION 2.5 [7]. *Let* $T=\{t_1, ..., t_s\} \subset K$ *and assume that* T *is an SWT-set w.r.t.* $U=\text{span}\{u_1, ..., u_n\}$. *Then for every* $\varepsilon>0$ *there exists a set* $\tilde{T}=\{\tilde{t}_1, ..., \tilde{t}_s\} \subset K$ *satisfying*

(i) $\tilde{T}$ *is an SWT-set w.r.t.* U;

(ii) *for every* $i \in \{1, ..., s\}$ *there exists* $j_i \in I$ *such that* $t_i \in K_{j_i}$ *and* $\tilde{t}_i \in \text{int}_K K_{j_i}$;

(iii) $\|t_i - \tilde{t}_i\| < \varepsilon$, $i=1, ..., s$.

3. INTERPOLATION BY MULTIVARIATE SPLINES

Assume now that in (2.1), $V_i = \Pi_m$, $i \in I$; i.e., S is a space of multivariate polynomial splines.

As an application of Theorem 2.3 we shall now show that every AI-set $T=\{t_1, ..., t_n\}$ can be changed to an I-set $\tilde{T}$ in a very general way. Indeed, let $T=\{t_1, ..., t_n\} \subset K$ being an SWT-set w.r.t. $U=\text{span}\{u_1, ..., u_n\}$ and let $\varepsilon>0$. By Proposition 2.5 there exists $\tilde{T}=\{\tilde{t}_1, ..., \tilde{t}_n\} \subset K$ satisfying

(i) $\tilde{T}$ is an SWT-set w.r.t. U;

(ii) for every $i \in \{1, ..., n\}$ there exists $j_i \in I$ such that $t_i \in K_{j_i}$ and $\tilde{t}_i \in \text{int}_K K_{j_i}$;

(iii) $\|t_i - \tilde{t}_i\| < \varepsilon$, $i=1, ..., n$.

To determine a direction for changing t_i choose any point $v_i \in \text{int}_K K_{j_i}$, $i = 1, ..., n$. Then in view of (i) and (ii) above, we already know that $V := \{v_1, ..., v_n\}$ is an SWT-set w.r.t.U. Therefore by Theorem 2.3, there exists an I-set $\tilde{V} := \{\tilde{v}_1, ..., \tilde{v}_n\}$ such that $\tilde{v}_i \in K_{j_i}$ and $\|v_i - \tilde{v}_i\| < \varepsilon$, $i = 1, ..., n$. Now considering the line through t_i and $\tilde{v}_i$ we define

$$l_i := \{t \in K_{j_i}: \text{there exists } \lambda \in \mathbb{R} \text{ such that } t = t_i(\lambda) = (1 - \lambda)\, t_i + \lambda \tilde{v}_i\},$$

$i = 1, ..., n$. In particular, since K_{j_i} is convex, $t_i(\lambda) \in l_i$ if $0 \leqslant \lambda \leqslant 1$. (In fact, l_i also depends on ε, but we may omit it, because ε will not be changed in the sequel.)

We are now ready to state a result on existence of I-sets on $\bigcup_{i=1}^n l_i$.

THEOREM 3.1. *Let the same hypotheses as above be given. Then there exists a real positive number λ_0 such that $\bar{T}$ defined by*

$$\bar{T} := \bar{T}(\lambda) := \{t_1(\lambda), ..., t_n(\lambda)\}$$

where $t_i(\lambda) \in l_i$, $i = 1, ..., n$ is an I-set w.r.t.U for every $0 < \lambda < \lambda_0$. If $t_i(\lambda) \in l_i$ for every $-\lambda_0 < \lambda < 0$ and every $i = 1, ..., n$, the statement is even true for $0 < |\lambda| < \lambda_0$. Moreover, if T is an I-set w.r.t.U, the statement also holds if $\lambda = 0$.

Proof. Let $\{u_1, ..., u_n\}$ be any basis of U. Since $\tilde{V}$ is an I-set w.r.t.U, it follows that

$$\det(u_i(\tilde{v}_j))_{i,j=1}^n \neq 0.$$

Set $D(\lambda) := \det(u_i(t_j(\lambda)))_{i,j=1}^n$, $0 \leqslant \lambda \leqslant 1$. Hence, $D(1) \neq 0$. Since every u_l is a polynomial of total degree at most m on K_{j_i}, $i = 1, ..., n$, it clearly follows that u_l is a polynomial in λ of degree at most m on l_i. This implies that $D(\cdot)$ is a polynomial in λ of degree at most nm, $0 \leqslant \lambda \leqslant 1$. Then it follows from $D(1) \neq 0$ that $D(\cdot)$ has at most finitely many zeros in $[0, 1)$.

Hence there exists $\lambda_0 > 0$ such that $D(\lambda) \neq 0$ for every $0 < \lambda < \lambda_0$.

If $t_i(\lambda) \in l_i$ for every $-\lambda_0 < \lambda < 0$ and every $i = 1, ..., n$, then defining $D(\lambda)$ for $-\lambda_0 < \lambda \leqslant 1$ and arguing as above we obtain $D(\lambda) \neq 0$ if $0 < |\lambda| < \lambda_0$.

Finally, if T is an I-set, then $D(0) \neq 0$, and the proof is completed. ∎

Remark 3.2. Since ε can be chosen as small as possible, the above statement shows that an AI-set T with $\text{card}(T) = n$ can be changed to an I-set shifting every element t_i of T in nearly every direction within the polyhedron K_{j_i}.

Sufficient conditions for a set T to be an I-set and algorithms for constructing I-sets are given in some special cases of multivariate spline interpolation (for references see [7]). In the case of bivariate linear splines such an algorithm was developed by Chui, He and Wang (see Chui [1], Chapter 9). Using the methods in the proof of Theorem 3.1 we shall now examine this case more detailed and shall describe a simple algorithm to construct a wide class of I-sets.

Let K denote a *regular triangulation* in $\mathbb{R}^2$; i.e., $K = \bigcup_{i \in I} K_i \subset \mathbb{R}^2$ where $\{K_i\}_{i \in I}$ is a set of triangles with the property that no vertex of K_i lies on the interior of a side of any other K_j $(i, j \in I)$. Assume that $\{e_1, ..., e_n\}$ denotes the set of all vertices of the triangles K_i in K $(i \in I)$. If U is the subspace of bivariate linear splines in $C(K)$, then it is well-known (see Chui [1], p. 136) that $\dim U = n$ and there exists a basis $\{u_1, ..., u_n\}$ defined uniquely by $u_i(e_j) := \delta_{ij}$, $i, j = 1, ..., n$. Of course each u_i is a minimally supported function in U. It is usually called a Courant (hat) function. We define

$$L_i := \{t \in K: u_i(t) > \tfrac{1}{2}\}, \qquad i = 1, ..., n.$$

If $t_i \in L_i$, then in view of $\sum_{j=1}^n u_j(t_i) = 1$ we have

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} u_j(t_i) = 1 - u_i(t_i) < \tfrac{1}{2}.$$

In addition, the left-sided sum contains at most two terms which are non-zero. Hence it follows that the matrix $(u_j(t_i))_{i,j=1}^n$ is diagonally dominant which implies that $\{t_1, ..., t_n\}$ is an I-set w.r.t. U for every choice of points $t_i \in L_i$, $i = 1, ..., n$.

Starting with an arbitrary AI-set we shall now apply this fact to construct I-sets as follows. Suppose that $T = \{t_1, ..., t_n\}$ is an AI-set w.r.t. U. Then by Theorem 2.3 and Proposition 2.4 there exists some permutation σ of $\{1, ..., n\}$ such that

$$t_{\sigma(i)} \in S_i = \operatorname{supp} u_i, \qquad i = 1, ..., n.$$

W.l.o.g. we may assume that $\sigma(i) = i$, $i = 1, ..., n$. Moreover, by definition of u_i we have that

$$S_i = \bigcup \{K_l: e_i \text{ is a vertex of } K_l\}.$$

This implies that $t_i \in K_{j_i} \subset S_i$ for some $j_i \in I$, $i = 1, ..., n$. Since $L_i \cap K_{j_i} \neq \emptyset$, we can choose any $\tilde{t}_i \in L_i \cap K_{j_i}$ and define

$$l_i := \{\bar{t}_i \in K_{j_i}: \text{there exists } \lambda \in \mathbb{R} \text{ such that } \bar{t}_i = \bar{t}_i(\lambda) = (1 - \lambda)\, t_i + \lambda \tilde{t}_i\},$$

$i = 1, ..., n$. In particular, it follows that $\bar{t}_i(\lambda) \in l_i$, $0 \leqslant \lambda \leqslant 1$.

We are now able to show that T can be easily changed to an I-set on $\bigcup_{i=1}^{n} l_i$.

THEOREM 3.3. *Let the same hypotheses as above be given. Then there exists a real positive number* λ_0 *such that*

$$\tilde{T} := \tilde{T}(\lambda) := \{\tilde{t}_1(\lambda), ..., \tilde{t}_n(\lambda)\}$$

where $\tilde{t}_i(\lambda) \in l_i$, $i = 1, ..., n$ *is an I-set w.r.t.* U *for every* $0 < \lambda < \lambda_0$. *If* $\tilde{t}_i(\lambda) \in l_i$ *for every* $-\lambda_0 < \lambda < 0$ *and every* $i = 1, ..., n$, *the statement is even true for* $0 < |\lambda| < \lambda_0$. *Moreover, if* T *is an I-set w.r.t.* U, *the statement also holds if* $\lambda = 0$.

Proof. The statement can be analogously proved as Theorem 3.1. ∎

Remark 3.4. (i) Let for $i \in \{1, ..., n\}$ K_{j_i} be any triangle in K such that e_i is a vertex of K_{j_i}. Assume that this triangle is defined by the vertices e_i, f_1, f_2. Set $m_l := (e_i + f_l)/2$, $l = 1, 2$, the midpoints of two sides in K_{j_i}, and draw a line g_i through m_1 and m_2. Since $u_i(e_i) = 1$ and $u_i(f_l) = 0$, $l = 1, 2$, it is obvious that $u_i(t) = \frac{1}{2}$ for every $t \in g_i \cap K_{j_i}$. Hence it follows that $L_i \cap K_{j_i}$ where L_i is defined as above is the triangle with the vertices e_i, m_1, m_2, except the side $g_i \cap K_{j_i}$.

This shows that each L_i can be easily determined.

(ii) The subsets $\{L_i\}_{i=1}^{n}$ of K are maximal in the sense that Theorem 3.3 is no longer true if L_i is extended to its closure

$$\bar{L}_i := \{t \in K: u_i(t) \geq \tfrac{1}{2}\}, \qquad i = 1, ..., n.$$

To show this let $e_1 = (0, 0)$, $e_2 = (1, 0)$, $e_3 = (1, 1)$, $e_4 = (0, 1) \in \mathbb{R}^2$, and let K_1, K_2 be triangles with the vertices e_1, e_3, e_4 and e_1, e_2, e_3, respectively. Let $K = K_1 \cup K_2 = [0, 1] \times [0, 1]$ and $T = \{t_1, ..., t_4\}$ where $t_1 = (\frac{5}{8}, \frac{1}{8})$, $t_2 = (\frac{7}{8}, \frac{3}{8})$, $t_3 = (\frac{3}{8}, \frac{7}{8})$, $t_4 = (\frac{1}{8}, \frac{5}{8})$. Assume that $U = \text{span}\{u_1, ..., u_4\}$ where $u_i(e_j) = \delta_{ij}$, $i, j = 1, ..., 4$. Then the function $u_0 \in U$ defined by

$$u_0(x, y) := \begin{cases} \frac{1}{2} + x - y, & \text{if } (x, y) \in K_1 \\ \frac{1}{2} - x + y, & \text{if } (x, y) \in K_2 \end{cases}$$

satisfies $u_0(t_i) = 0$, $i = 1, ..., 4$. Moreover, we obtain $\bar{L}_1 = K \cap \{(x, y): x + y - \frac{1}{2} \leq 0\}$, $\bar{L}_2 = K \cap \{(x, y): x - y - \frac{1}{2} \geq 0\}$, $\bar{L}_3 = K \cap \{(x, y): x + y - \frac{3}{2} \geq 0\}$, $\bar{L}_4 = K \cap \{(x, y): x - y + \frac{1}{2} \leq 0\}$. If we define $\tilde{t}_i \in \bar{L}_i$, $i = 1, ..., 4$ by $\tilde{t}_1 = (\frac{1}{2}, 0)$, $\tilde{t}_2 = (1, \frac{1}{2})$, $\tilde{t}_3 = (\frac{1}{2}, 1)$, $\tilde{t}_4 = (0, \frac{1}{2})$, we then obtain $u_0(\tilde{t}_i) = 0$ which shows that $u_0(t) = 0$, $t \in [t_i, \tilde{t}_i]$, $i = 1, ..., 4$. Therefore, T cannot be changed to an I-set w.r.t. U when shifting T on the straight lines through t_i and $\tilde{t}_i$, $i = 1, ..., 4$.

4. Interpolation by Generalized Splines

Throughout Sections 4–6 we shall assume that $K = [a, b] \subset \mathbb{R}$. Hence by definition of K there exists a knot partition Δ: $a = x_0 < x_1 < \cdots < x_{r+1} = b$ $(r \geq 0)$ such that $K_i = [x_i, x_{i+1}]$, $i = 0, ..., r$ and

$$K = \bigcup_{i=0}^{r} K_i = [a, b].$$

Associated with the partition Δ we consider finite-dimensional linear subspaces U of $C[a, b]$ such that for each $i \in \{0, ..., r\}$ the space $U_i := U|_{K_i}$ has the *(NV)-property*: If $u \in U_i \backslash \{0\}$, then u does not vanish identically on any subinterval of K_i. (Here and in the sequel a subinterval I is always assumed to be nondegenerate; i.e., $I = [\alpha, \beta]$ where $\alpha < \beta$.)

Note that the most important examples of spaces U_i with the (NV)-property are the Haar subspaces of $C(K_i)$.

Thus associated with the partition Δ we consider linear subspaces

$$U := U(\Delta) := \{u \in C[a, b]: U_i := U|_{K_i} \text{ has the (NV)-property, } i = 0, ..., r\}. \tag{4.1}$$

In [3], a special class of such spaces U was introduced. U was defined there by Haar subspaces U_i of $C(K_i)$, $i = 0, ..., r$ and by linear functionals describing how the ith and the jth pieces $u|_{K_i}$ and $u|_{K_j}$, respectively, of the functions $u \in U$ are tied together. Therefore, in analogy to [3] we call every U defined as in (4.1) a *space of generalized splines, associated with Δ and $U_0, ..., U_r$*. We set

$$GS_n := \{U \subset C[a, b]: \dim U = n, \ U \text{ is defined as in (4.1)}\} \tag{4.2}$$

and call GS_n the class of *generalized spline spaces.*

Let $U \in GS_n$ and $T = \{t_1, ..., t_s\} \subset (a, b)$ such that $s \leq n$ and $T \cap Z(U) = \varnothing$ where

$$Z(U) := \{t \in [a, b]: u(t) = 0 \text{ for every } u \in U\}.$$

We are interested in a necessary and sufficient condition for T ensuring $\dim U|_T = s$. (The assumption $T \subset (a, b)$ cannot be omitted as we shall show in Remark 4.11.)

In view of Theorem 2.3, we already know that in the case when $s = n$ the set T is an AI-set w.r.t. U if and only if T is an SWT-set. Hence it seems to be natural to consider subsets T of $[a, b]$ satisfying a slightly stronger condition.

DEFINITION 4.1. Let $U \in GS_n$ and $T = \{t_1, ..., t_s\} \subset (a, b)\backslash Z(U)$. Then we say that T satisfies a *strong condition of Schoenberg–Whitney type* or T is an *SSW-set w.r.t. U* if

$$\text{card}(T \cap R) \leqslant \dim U|_R$$

for every choice of subsets $R := \bigcup_{j=1}^{l} K_{i_j}$ of $[a, b]$ where $0 \leqslant i_1 < \cdots < i_l \leqslant r$.

This condition is obviously necessary for T to be an I-set w.r.t. U.

LEMMA 4.2. *If T is an I-set w.r.t. U, then T is an SSW-set w.r.t. U.*

Proof. Suppose that T is an I-set but fails to be an SSW-set. Hence

$$c := \text{card}(T \cap R) > \dim U|_R := \tilde{c}$$

for some $R := \bigcup_{j=1}^{l} K_{i_j}$. Thus we could interpolate arbitrary data $\{y_1, ..., y_c\}$ by $U|_R$ at $T \cap R$ which contradicts $c > \tilde{c}$. ∎

Remark 4.3. A simple example shows that the converse of the above statement is not true in general:

Let $K = [-1, 1]$ and $-1 = x_0 < x_1 = 1$. Assume that $U = \text{span}\{u_1, u_2\}$ where $u_1 \equiv 1$ and $u_2(t) := t^2$, $t \in K$. It then follows immediately that T is an SSW-set w.r.t. U whenever $T = \{t_1, t_2\}$ and $-1 < t_1 < t_2 < 1$. On the other hand, the function $\alpha^2 u_1 - u_2$ has the zeros $-\alpha, \alpha$ where $0 < \alpha < 1$ which implies that $T_\alpha := \{-\alpha, \alpha\}$ fails to be an I-set.

This remark leads us to make the following definition.

DEFINITION 4.4. Let $U \in GS_n$. Then U is said to have the *SSW-property* (respectively the *SSW_n-property*) if every SSW-set T w.r.t. U (respectively every SSW-set T w.r.t. U such that $\text{card}(T) = n$) is an I-set.

Moreover, U is said to have the *weak SSW-property* if U has the SSW_n-property and every SSW-set T w.r.t. U such that $\text{card}(T) < n$ and $T \subset \bigcup_{i=0}^{r} \text{int}_K K_i$ is an I-set.

We are now interested in which generalized spline spaces possess these interpolation properties. It turns out that in this context the class of weak Chebyshev spaces plays an important role.

DEFINITION 4.5. An n-dimensional subspace U of $C[a, b]$ is called a *weak Chebyshev* or *WT-space* if every $u \in U$ has at most $n - 1$ sign changes; i.e., there do not exist points $a \leqslant z_1 < \cdots < z_{n+1} \leqslant b$ such that

$$u(z_i)\, u(z_{i+1}) < 0, \qquad i = 1, ..., n.$$

We define

$$WT_n := \{U \subset C[a, b] \colon \dim U = n,\ U \text{ is a WT-space}\}$$

and are now ready to state the main results of this section.

THEOREM 4.6. *Let $U \in GS_n$ and assume that $U \in WT_n$. Then U has the weak SSW-property.*

The proof of this statement will be given in Section 6. We now show by an example that in the preceding theorem the weak SSW-property cannot be replaced by the SSW-property.

EXAMPLE 4.7. Let $K = [-2, 2]$ and $K_i = [x_i, x_{i+1}]$ where $x_i = i - 2$, $i = 0, \ldots, 4$. Suppose that $U = \operatorname{span}\{u_1, \ldots, u_4\}$ where $u_1(t) := t$, $t \in K$,

$$u_2(t) := \begin{cases} 0, & \text{if } t \in [-2, 1] \\ t - 1, & \text{if } t \in [1, 2], \end{cases}$$

$u_3(t) := u_2(-t)$, $t \in K$ and

$$u_4(t) := \begin{cases} 0, & \text{if } t \in [-2, -1] \cup [1, 2] \\ 1 - t^2, & \text{if } t \in [-1, 1]. \end{cases}$$

It is easily verified that $U \in GS_4 \cap WT_4$ and $Z(U) = \varnothing$. Let $T := \{-3/2, -1, 1\}$. Then it is easily seen that T is an SSW-set w.r.t. U. But T fails to be an I-set, since $\dim U|_T = 2 < \operatorname{card}(T)$.

This shows that U does not have the SSW-property.

We now show by a simple example that the converse of Theorem 4.6 is not true.

EXAMPLE 4.8. Let $K = [-1, 1]$, $x_0 = -1$, $x_1 = 1$ and $U = \operatorname{span}\{u_1\}$ where $u_1(t) := t$, $t \in [-1, 1]$. If $T = \{t_1\} \subset (-1, 1) \setminus \{0\}$, then T is trivially an SSW-set w.r.t. U. Moreover, $u_1(t_1) \neq 0$ which implies that U has the SSW-property. But $U \notin WT_1$, because u_1 has a sign change at $t = 0$.

We shall now show that the converse of Theorem 4.6 is true under weak additional assumptions on U. Let $A \subset \mathbb{R}$ and $F(A)$ denote the linear space of all real valued functions on A. Following [2] we call a finite-dimensional subspace U of $F(A)$ *regular* if from the conditions $u \in U$, $u(t_1)\, u_2(t_2) < 0$ where $t_1, t_2 \in A$, $t_1 < t_2$ it follows that there exists $t \in A \setminus Z(U)$ such that $t_1 < t < t_2$ and $u(t) = 0$. (In particular, U is regular if $A = [a, b]$, $U \subset C[a, b]$ and $Z(U) \cap (a, b) = \varnothing$.)

THEOREM 4.9. *Let $U \in GS_n$ and assume that U is regular. The following conditions are equivalent.*

(i) *U has the weak SSW-property;*

(ii) *U has the SSW_n-property;*

(iii) *$U \in WT_n$.*

The proof will also be given in Section 6. As a consequence of Theorem 4.9 we obtain a statement on the restrictions of U to the knot intervals K_i.

COROLLARY 4.10. *Assume that $U \in GS_n$ and has the SSW_n-property. Moreover, assume that U is regular. Then for every $i \in \{0, ..., r\}$ U has the Haar property both in $[x_i, x_{i+1})\backslash Z(U)$ and in $(x_i, x_{i+1}]\backslash Z(U)$.*

Proof. It follows from Theorem 4.9 that $U \in WT_n$. Then by a result in [6] $U|_{K_i}$ is a WT-space of dimension n_i, $i = 0, ..., r$. Suppose now that for some $i \in \{0, ..., r\}$ U fails to have the Haar property in $[x_i, x_{i+1})\backslash Z(U)$. Therefore, and by (4.1) there must exist $\tilde{u} \in U_i\backslash\{0\}$ and points $x_i \leqslant z_1 < y_1 < z_2 < \cdots < y_{n_i-1} < z_{n_i} < x_{i+1}$ such that

(i) $\tilde{u}(z_i) = 0$, $i = 1, ..., n_i$;

(ii) $\tilde{u}(y_i) \neq 0$, $i = 1, ..., n_i - 1$;

(iii) $\{z_1, ..., z_{n_i}\} \cap Z(U) = \emptyset$.

Then from a result of Stockenberg ([8] or Theorem 2.45 of [5]) it follows that $\tilde{u}(t) = 0$ for every $t \in [z_{n_i}, x_{i+1}]$, a contradiction of the assumption on U_i. ∎

Remark 4.11. In the above statements we consider subsets $T = \{t_1, ..., t_s\}$ satisfying $T \cap Z(U) = \emptyset$ and $T \subset (a, b)$. While the first assumption is trivially necessary for T to be an I-set, the assumption $T \subset (a, b)$ seems to be unnecessary. But this is not true as the following example shows:

Let $K = [-2, 1]$ and $K_i = [x_i, x_{i+1}]$ where $x_i = i - 2$, $i = 0, 1, 2, 3$. Suppose that $U = \text{span}\{u_1, u_2\}$ where $u_1(t) := t$, $t \in [-2, 1]$ and

$$u_2(t) := \begin{cases} 0, & \text{if } t \in [-2, -1] \\ 1 - t^2, & \text{if } t \in [-1, 1]. \end{cases}$$

It follows immediately that $U \in GS_2 \cap WT_2$ and $U|_{K_i}$ is a Haar space, $i = 0, 1, 2$. Moreover, $Z(U) = \emptyset$. Let $T := \{-1, 1\}$. Then it is easily verified that

$$\text{card}(T \cap R) \leqslant \dim U|_R$$

for every $R := \bigcup_{j=1}^{l} K_{i_j} \subset [-2, 1]$. Hence T has the property of an SSW-set w.r.t. U. But T fails to be an I-set, since $u_2(-1) = u_2(1) = 0$.

This shows that Theorem 4.9 cannot be extended in the sense that U has the SSW_n-property for every $T \subset [a, b)$ or $T \subset (a, b]$, respectively.

5. The Interlacing and (SW)-Property

Interpolation by generalized splines and a bigger subclass of weak Chebyshev spaces was treated in [3] and by Davydov [2], respectively. In this section we shall compare the results there with Theorem 4.9.

The interlacing property. Let $\Delta: a = x_0 < \cdots < x_{r+1} = b$ and $K_i := [x_i, x_{i+1}]$, $i = 0, ..., r$. For every $i \in \{0, ..., r\}$ suppose that U_i is a Haar subspace of $C(K_i)$ of dimension $n_i \geqslant 1$. Moreover, suppose that

$$\Gamma := \{\Gamma_{ij} : 0 \leqslant i < j \leqslant r\}, \qquad \Gamma_{ij} := \{(\underline{\gamma}_v^{ij}, \bar{\gamma}_v^{ij})\}_{v=1}^{r_{ij}},$$

where the $\underline{\gamma}_v^{ij}$ and $\bar{\gamma}_v^{ij}$ are linear functionals defined on U_i and U_j, respectively. In [3] a generalized spline space S was defined by

$$\begin{aligned} S &:= S(U_0, ..., U_r; \Gamma; \Delta) \\ &:= \{s \in C[a, b] : s_i = s|_{K_i} \in U_i,\ i = 0, ..., r \text{ and} \\ &\quad \underline{\gamma}_v^{ij} s_i = \bar{\gamma}_v^{ij} s_j,\ v = 1, ..., r_{ij},\ 0 \leqslant i < j \leqslant r\}. \end{aligned} \tag{5.1}$$

Comparing (5.1) with (4.2) we see that

$$S \in GS_n, \qquad \text{if} \quad \dim S = n.$$

Suppose that S is defined as in (5.1) and $\dim S = n$. To formulate the interlacing property we need the notation

$$n_{ij} := \dim S|_{[x_i, x_j]}, \qquad 0 \leqslant i < j \leqslant r + 1.$$

Definition 5.1 [3]. S is said to possess the *interlacing property* provided a set $T = \{t_1, ..., t_n\}$ where $a \leqslant t_1 < \cdots < t_n \leqslant b$ is an I-set w.r.t. S if and only if it satisfies the condition

$$t_{n - n_{i,r+1}} < x_i < t_{n_{0,i}+1}, \qquad i = 1, ..., r. \tag{5.2}$$

By Theorem 2.5 in [3] a characterization of which generalized spline spaces $S = S(U_0, ..., U_r; \Gamma, \Delta)$ have the interlacing property is given. Let us denote all the n-dimensional spaces S with this property by

$$IP_n.$$

In particular, from the results in [3] it follows that

$$IP_n \subsetneq WT_n.$$

An important subclass of IP_n forms the class of polynomial spline spaces $S_m(\Delta)$ of degree m with r fixed knots. Indeed, condition (5.2) is derived from the classical Schoenberg–Whitney condition [4]

$$t_i < x_i < t_{i+m+1}, \qquad i = 1, ..., r$$

which characterizes every I-set $T = \{t_1, ..., t_{m+r+1}\}$ w.r.t. $S_m(\Delta)$.

The (SW)-property. Since there exist generalized spline spaces S defined as in (5.1) which are not contained in IP_n (see Section 4 in [3] and Remark 2.2 (ii) in [7]), in [2] a more general condition ensuring unique Lagrange interpolation was introduced. To formulate it let $A \subset \mathbb{R}$ and $F(A)$ denote the linear space of all real valued functions on A. Suppose that U is a finite-dimensional subspace of $F(A)$.

DEFINITION 5.2 [2]. U is said to possess the *(SW)-property* provided the condition

$$\operatorname{card}(M \cap [\alpha, \beta]) \leqslant \dim U|_{A \cap [\alpha, \beta]} \tag{5.3}$$

for all $\alpha, \beta \in \mathbb{R}$, $\alpha \leqslant \beta$ is necessary and sufficient for every I-set $M \subset A$ w.r.t. U.

THEOREM 5.3 [2]. *The following conditions are equivalent.*

(i) *U has the (SW)-property and is a weak Chebyshev subspace of $F(A)$;*

(ii) *$U|_{\tilde{A}}$ is a weak Chebyshev space for every subset $\tilde{A}$ of A.*

In the case of regular subspaces of $F(A)$ (for definition see Section 4) the (SW)-property can be even characterized by statement (ii) of Theorem 5.3.

THEOREM 5.4 [2]. *Let U be regular. Then the following conditions are equivalent.*

(i) *U has the (SW)-property;*

(ii) *$U|_{\tilde{A}}$ is a weak Chebyshev space for every subset $\tilde{A}$ of A.*

It is noted in [2] that if $A=[a,b]$ and $U\in IP_n$, then U has the (SW)-property. The converse however is not true as we have shown in [7], Remark 2.2 (ii), (iv) by a simple example of a generalized spline space U which has the (SW)-property but fails to have the interlacing property.

The weak SSW-property. Theorem 4.9 states that for a regular generalized spline space U the weak SSW-property can be characterized by the weak Chebyshev property. Moreover, it follows from Definition 5.2 that for spaces of generalized splines the (SW)-property clearly implies the weak SSW-property. Hence, in view of Theorem 5.4, one could ask the question of whether for regular generalized spline spaces (SW)-property and weak SSW-property are equivalent. This is not true as the following example shows.

EXAMPLE 5.5. Let $K=[-2,2]$ and $K_i=[x_i,x_{i+1}]$ where $x_i=i-2$, $i=0,...,4$. Suppose $U_0=U_3=\operatorname{span}\{t\}$ while $U_1=U_2=\operatorname{span}\{t,1-t^2\}$. Hence U_i is a Haar subspace of $C(K_i)$, $i=0,...,3$. We consider the generalized spline space

$$U=\{u\in C[-2,2]\colon u_i=u|_{K_i}\in U_i,\ i=0,...,3 \text{ and } D_-u(0)=D_+u(0)\}$$

where D_- and D_+ denote the left- and right-sided derivative of u, respectively. This implies that $U=\operatorname{span}\{u_1,u_2\}$ where $u_1(t):=t$, $t\in[-2,2]$ and

$$u_2(t):=\begin{cases}0, & \text{if } t\in[-2,-1]\cup[1,2]\\ 1-t^2, & \text{if } t\in[-1,1].\end{cases}$$

It follows immediately that $U\in GS_2\cap WT_2$. Hence by Theorem 4.6, U has the weak SSW-property.

But in view of Theorem 5.3, U fails to have the (SW)-property, because $U|_{K_0\cup K_3}$ is spanned by the function u_1 which changes the sign on $K_0\cup K_3$. The last statement can be also obtained directly: Let $T=\{t_1,t_2\}$ where $t_1=-1$, $t_2=1$. Then

$$\operatorname{card}(T\cap[\alpha,\beta])\leqslant 1=\dim U|_{[\alpha,\beta]},$$

if $[\alpha,\beta]\subset[-2,-1]\cup[1,2]$, and

$$\operatorname{card}(T\cap[\alpha,\beta])\leqslant 2=\dim U|_{[\alpha,\beta]},$$

if $[\alpha,\beta]\cap(-1,1)\neq\varnothing$. This shows that T satisfies (5.3). On the other hand, $u_2(t_i)=0$, $i=1,2$ which implies that U fails to have the (SW)-property.

6. PROOFS

Proof of Theorem 4.6. Let $U \in WT_n$. We first show the following statement.

Claim 1. U has the SSW_n-property.

Proof. Assume that $T = \{t_1, ..., t_n\} \subset (a, b) \backslash Z(U)$ such that T is an SSW-set w.r.t. U. We have to show that T is an I-set.

Since $T \subset (a, b) \backslash Z(U)$, it is no restriction to assume that no knot interval K_i is contained in $Z(U)$.

Assume that T fails to be an I-set w.r.t. U. Hence there exists $u_0 \in U \backslash \{0\}$ such that $u_0(t_i) = 0$, $i = 1, ..., n$. We consider two cases.

Case 1. Suppose that u_0 does not vanish identically on a knot interval K_j. Then there must exist $z_i \in (t_i, t_{i+1})$ such that $u_0(z_i) \neq 0$, $i = 1, ..., n-1$. Moreover, since $T \cap Z(U) = \emptyset$, for every $i \in \{1, ..., n\}$ there exists $u_i \in U$ satisfying $u_i(t_i) \neq 0$. Then by a result of Stockenberg ([8] or Theorem 2.45 of [5]) we must have $t_1 = a$ and $t_n = b$, a contradiction to the choice of T.

Case 2. Suppose that $u_0 \equiv 0$ on $R := \bigcup_{j=1}^{l} K_{i_j} \subset [a, b]$ and does not vanish identically on a knot interval outside of R. Hence there exists $j \in \{1, ..., l\}$ such that $u_0 \not\equiv 0$ in $K_{i_j - 1}$ or in $K_{i_j + 1}$. W.l.o.g. we may assume that $u_0 \not\equiv 0$ in $K_{i_j - 1}$. (In the case when $u_0 \equiv 0$ in $K_{i_j - 1}$ and $u_0 \not\equiv 0$ in $K_{i_j + 1}$, all the following arguments for the left side of K_{i_j} must be analogously applied to the right side of K_{i_j}.) To simplify our notations we set $i_j = i$ and define $c_1 := \dim U|_{K_i}$. Then $u_0 \equiv 0$ in K_i and $u_0 \not\equiv 0$ in K_{i-1} where $K_i = [x_i, x_{i+1}]$. Using arguments from linear Algebra we find points $x_i < w_1 < \cdots < w_{c_1} < x_{i+1}$ such that

$$\det(u_i(w_j))_{i,j=1}^{c_1} \neq 0$$

where $u_1, ..., u_{c_1}$ are functions in U which are linearly independent in K_i. We extend this point set by a point set $\{w_{c_1+1}, ..., w_c\}$ from $(a, b) \cap (R \backslash K_i)$ where $c := \dim U|_R$ such that $\dim U|_{\{w_1, ..., w_c\}} = c$. This also follows from linear Algebra and the obvious fact that $\dim U|_R = \dim U|_{K_i} + \dim \tilde{U}|_R$ where $\tilde{U} := \{u \in U: u \equiv 0 \text{ in } K_i\}$.

Since $u_0 \not\equiv 0$ in K_{i-1}, there exists a sequence $(y_m) \subset K_{i-1}$ converging to x_i such that

$$\operatorname{sign} u_0(y_m) =: \sigma$$

for every $m \in \mathbb{N}$ where $\sigma \in \{-1, 1\}$ independently of m. By the choice of $\{w_1, ..., w_c\}$ there exists a $v_1 \in U$ such that

$$v_1(w_i) = \sigma(-1)^{i-1}, \qquad i = 1, ..., c_1,$$

$$v_1(w_i) = 0, \qquad i = c_1 + 1, ..., c.$$

Since by assumption $U \in WT_n$, it follows from a result in [6] that $U|_{K_i} \in WT_{c_1}$. Hence this implies that $\sigma v_1(x_i) \geq 0$.

Let $\tilde{R} := (a, b) \backslash R$ and $\tilde{T} := T \cap \tilde{R}$. Recall that $\tilde{T} \subset Z(u_0)$ and u_0 does not vanish identically on any subinterval of $\tilde{R}$. Hence if $\tilde{t} \in \tilde{T}$, then in every neighborhood of $\tilde{t}$ there exists $w \in \tilde{R}$ such that $u_0(w) \neq 0$. Moreover, recall that T is assumed to be an SSW-set w.r.t. U. Hence,

$$\text{card}(T \cap R) \leq \dim U|_R = c$$

which implies that $\text{card}(\tilde{T}) \geq n - c$. We now classify the set $\tilde{T}$ as follows:

$Z_{sc} := \{t \in \tilde{T}: u_0$ changes the sign at t or $t \in Z(v_1)\}$;

$Z_+ := \{t \in \tilde{T}: t \notin Z_{sc}$ and $u_0 v_1 \geq 0$ in some neighborhood $(t - \delta, t + \delta)$ of $t\}$;

$Z_- := \{t \in \tilde{T}: t \notin Z_{sc}$ and $u_0 v_1 \leq 0$ in some neighborhood $(t - \delta, t + \delta)$ of $t\}$.

Then $\tilde{T} = Z_{sc} \cup Z_+ \cup Z_-$ and $Z_+ \cap Z_- = \varnothing$. We consider three more cases.

Case 2 (a). Suppose that $\text{card}(Z_+) > \text{card}(Z_-)$. Then for every sufficiently small $\varepsilon > 0$ the function $u_0 - \varepsilon v_1$ has at least two sign changes in a small neighborhood of t for every $t \in Z_+$ and a zero at t or in a small neighborhood of t for every $t \in Z_{sc}$ (even a sign change if u_0 changes the sign at t). Since $Z_+ \cup Z_{sc} \subset \tilde{R}$ and $\tilde{R}$ is open in K, we may assume that all these zeros are also contained in $\tilde{R}$. Moreover, $u_0 - \varepsilon v_1$ has $c - c_1$ zeros $\{w_{c_1+1}, ..., w_c\} \subset R \backslash K_i$. Since $(T \cup \{w_1, ..., w_c\}) \cap Z(U) = \varnothing$, it easily follows that all these zeros of $u_0 - \varepsilon v_1$ can be obtained in $(a, b) \backslash Z(U)$. In addition, $u_0 - \varepsilon v_1$ has $c_1 - 1$ zeros with sign changes in K_i (where some of them could be elements of $Z(U)$) and, since ε is sufficiently small, $u_0 - \varepsilon v_1$ does not vanish identically on a subinterval of $[a, b] \backslash (R \backslash K_i)$.

Thus we can obtain a function with at least

$$\begin{aligned}
&2\,\text{card}(Z_+) + \text{card}(Z_{sc}) + c_1 - 1 + c - c_1 \\
&\quad \geq \text{card}(Z_+) + \text{card}(Z_-) + \text{card}(Z_{sc}) + c \\
&\quad = \text{card}(\tilde{T}) + c \geq n - c + c = n
\end{aligned}$$

zeros in (a, b).

Case 2 (b). Suppose that $\mathrm{card}(Z_-) > \mathrm{card}(Z_+)$. We then consider $u_0 + \varepsilon v_1$ where ε is a sufficiently small positive number and conclude analogously as in Case 2 (a).

Case 2 (c). Suppose that $\mathrm{card}(Z_+) = \mathrm{card}(Z_-)$. We consider the function $u_0 - \varepsilon v_1$ where ε is a sufficiently small positive number. Then this function has a zero with sign change at x_i or in some small left-sided neighborhood $(x_i - \delta, x_i)$. Moreover, arguing as in Case 2 (a) we have that $u_0 - \varepsilon v_1$ has at least $2\,\mathrm{card}(Z_+) + \mathrm{card}(Z_{sc})$ zeros in $\tilde{R} \cap ((a, b) \backslash Z(U))$, $c - c_1$ zeros $\{w_{c_1+1}, ..., w_c\} \subset (R \backslash K_i) \backslash Z(U)$, and $c_1 - 1$ zeros with sign changes in K_i. Summarizing we obtain

$$2\,\mathrm{card}(Z_+) + \mathrm{card}(Z_{sc}) + 1 + c - c_1 + c_1 - 1 = \mathrm{card}(\tilde{T}) + c \geqslant n - c + c = n$$

zeros in (a, b). In addition, since ε is sufficiently small, $u_0 - \varepsilon v_1$ does not vanish identically on a subinterval of $[a, b] \backslash (R \backslash K_i)$.

Summary Case 2. We have obtained a function $u_1 := u_0 \pm \varepsilon v_1$ satisfying the following properties:

(i) If $u_1 \equiv 0$ in some $K_j \subset [a, b]$, then $K_j \subset R_1 := R \backslash K_i$;

(ii) u_1 has at least $\mathrm{card}(\tilde{T}) + c_1$ zeros in $\tilde{R}_1 := (a, b) \backslash R_1$;

(iii) u_1 has $c - c_1$ zeros $\{w_{c_1+1}, ..., w_c\} \subset R_1 \backslash Z(U)$.

Hence, since $\mathrm{card}(\tilde{T}) \geqslant n - c$, there exist zeros $z_1 < \cdots < z_{n-c+c_1}$ of u_1 in $\tilde{R}_1$ with all the additional properties given in the Cases 2 (a)–(c). In particular, it then follows that $z_i \notin Z(U)$, if u_1 does not change the sign at z_i.

Let $T_1 := \{z_1, ..., z_{n-c+c_1}, w_{c_1+1}, ..., w_c\}$. Then $\mathrm{card}(T_1) = n$ and in view of the above properties, it is easily seen that

$$\mathrm{card}(T_1 \cap R_1) = c - c_1 = \dim U|_R - \dim U|_{K_i} \leqslant \dim U|_{R_1}.$$

Hence replacing u_0, T and R by u_1, T_1 and R_1, respectively we can again apply the methods of Case 1 or Case 2 and finally obtain a function $\tilde{u} \in U$ such that $\tilde{u}$ does not vanish identically on some K_j, $j = 0, ..., r$ and $\tilde{u}$ has n zeros $a < \tilde{z}_1 < \cdots < \tilde{z}_n < b$ with the additional property that each zero $\tilde{z}_i$ which fails to be a sign change of $\tilde{u}$ is an element of $(a, b) \backslash Z(U)$. Let $Z := \{\tilde{z}_1, ..., \tilde{z}_n\}$ and

$$Z_+(\tilde{u}) := \{\tilde{z}_i \in Z \colon \tilde{u} \text{ is nonnegative in some neighborhood of } \tilde{z}_i\};$$

$$Z_-(\tilde{u}) := \{\tilde{z}_i \in Z \colon \tilde{u} \text{ is nonpositive in some neighborhood of } \tilde{z}_i\};$$

$$Z_{sc}(\tilde{u}) := \{\tilde{z}_i \in Z \colon \tilde{u} \text{ changes the sign at } \tilde{z}_i\}.$$

If $Z_+(\tilde{u}) \cup Z_-(\tilde{u}) = \varnothing$, then $\tilde{u}$ would have n sign changes, contradicting the assumption $U \in WT_n$. Hence assume that $\mathrm{card}(Z_+(\tilde{u})) \geqslant \mathrm{card}(Z_-(\tilde{u}))$

and $Z_+(\tilde{u}) \neq \emptyset$. Let $z \in Z_+(\tilde{u})$. Since $z \notin Z(U)$, there exists $v_0 \in U$ such that $v_0(z) > 0$. Then arguing as in Case 2 we obtain a function $v_1 := \tilde{u} - \varepsilon v_0$ ($\varepsilon > 0$ sufficiently small) such that v_1 does not vanish identically on a subinterval of $[a, b]$, has at least n zeros in (a, b), $(Z_+(v_1) \cup Z_-(v_1)) \cap Z(U) = \emptyset$ and $\operatorname{card}(Z_{sc}(v_1)) \geqslant \operatorname{card}(Z_{sc}(\tilde{u})) + 1$ (where $Z_+(v_1)$, $Z_-(v_1)$, $Z_{sc}(v_1)$ are sets of zeros of v_1 analogously defined as for the function $\tilde{u}$).

Thus it is clear that after a finite number of steps we obtain a function $\tilde{v} \in U$ with at least n sign changes, a contradiction to the assumption $U \in WT_n$.

This completes the proof of Claim 1. ▌

Now we show that U has the weak SSW-property which will complete the proof of Theorem 4.6.

Claim 2. U has the weak SSW-property.

Proof. In view of Claim 1, we have still to show that every SSW-set $T \subset (a, b) \backslash Z(U)$ w.r.t. U such that $\operatorname{card}(T) = s < n$ and $T \subset \bigcup_{i=0}^{r} \operatorname{int}_K K_i$ is an I-set w.r.t. U.

Suppose that such a set T is given and assume that T fails to be an I-set. This means that

$$\dim U|_T < s.$$

We now construct an SSW-set $\hat{T}$ w.r.t. U such that $T \subset \hat{T}$ and $\operatorname{card}(\hat{T}) = s + 1$. Then it is obvious that

$$\dim U|_{\hat{T}} < s + 1$$

which implies that $\hat{T}$ fails to be an I-set. Applying this method $n - s$ times we finally shall arrive at an SSW-set $\tilde{T}$ such that $\operatorname{card}(\tilde{T}) = n$ and

$$\dim U|_{\tilde{T}} < n,$$

a contradiction to Claim 1.

Recall that T is assumed to be an SSW-set w.r.t. U and $T \subset \bigcup_{i=0}^{r} (x_i, x_{i+1})$. We consider two cases.

Case 1. Suppose that

$$\operatorname{card}(T \cap R) < \dim U|_R$$

for every $R := \bigcup_{j=1}^{l} K_{i_j} \subset K$. Then choosing any point $\hat{t} \in (a, b) \backslash (Z(U) \cup T \cup \{x_1, ..., x_r\})$ we define

$$\hat{T} := T \cup \{\hat{t}\}$$

and clearly obtain an SSW-set w.r.t. U.

Case 2. Suppose that

$$\operatorname{card}(T \cap R) = \dim U|_R$$

for some $R := \bigcup_{j=1}^{l} K_{i_j} \subset K$ and assume that R is maximal in the sense that

$$\operatorname{card}(T \cap \tilde{R}) < \dim U|_{\tilde{R}}$$

for every $\tilde{R} := \bigcup_{j=1}^{\tilde{l}} K_{m_j} \subset K$ where $\tilde{l} > l$.

Since $\operatorname{card}(T) < n$, it follows that $(x_i, x_{i+1}) \cap R = \varnothing$ for some $i \in \{0, ..., r\} \setminus \{i_1, ..., i_l\}$. Choosing any point $\tilde{t} \in (x_i, x_{i+1})$ we define

$$\hat{T} := T \cup \{\tilde{t}\}.$$

Let $\hat{R} := \bigcup_{j=1}^{k} K_{m_j}$. To show that

$$\operatorname{card}(\hat{T} \cap \hat{R}) \leqslant \dim U|_{\hat{R}} \tag{6.1}$$

we consider the following cases.

(i) Suppose that $\hat{R} \subset R$. Since $\hat{T} \cap \hat{R} = T \cap \hat{R}$ and $\operatorname{card}(T \cap \hat{R}) \leqslant \dim U|_{\hat{R}}$, (6.1) follows immediately.

(ii) Suppose that $\operatorname{int}_K \hat{R} \cap R = \varnothing$. We define

$$\tilde{R} := \hat{R} \cup R$$

and

$$\hat{U} := \{u \in U \colon u(t) = 0 \text{ for every } t \in R\}.$$

We then have that $\dim U|_{\tilde{R}} = \dim U|_R + \dim \hat{U}|_{\hat{R}}$. Since by definition,

$$T \subset \bigcup_{i=0}^{r} (x_i, x_{i+1}),$$

it follows that

$$\operatorname{card}(T \cap \tilde{R}) = \operatorname{card}(T \cap R) + \operatorname{card}(T \cap \hat{R}).$$

In view of the maximality of R, this implies that

$$\begin{aligned}\operatorname{card}(T \cap R) + \operatorname{card}(T \cap \hat{R}) &= \dim U|_R + \operatorname{card}(T \cap \hat{R}) \\ &< \dim U|_{\tilde{R}} = \dim U|_R + \dim \hat{U}|_{\hat{R}}.\end{aligned}$$

Hence we obtain that

$$\operatorname{card}(T \cap \hat{R}) < \dim \hat{U}|_{\hat{R}} \leqslant \dim U|_{\hat{R}}.$$

This proves (6.1).

(iii) Suppose that $\hat{R} = R_1 \cup R_2$ where $R_1 \subset R$ and $\text{int}_K R_2 \cap R = \varnothing$. It follows from (i) and (ii) that

$$\begin{aligned}\text{card}(\hat{T} \cap \hat{R}) &= \text{card}(T \cap R_1) + \text{card}(\hat{T} \cap R_2)\\ &\leqslant \dim U|_{R_1} + \dim \hat{U}|_{R_2} \leqslant \dim U|_{\hat{R}}.\end{aligned}$$

This proves (6.1).

Thus we have shown that $\hat{T}$ is an SSW-set w.r.t. U where $\text{card}(\hat{T}) = s+1$ and $\hat{T} \subset \bigcup_{i=0}^{r} \text{int}_K K_i$. This completes the proof of Claim 2. ∎

Proof of Theorem 4.9. Assume that U is regular. In view of Theorem 4.6, we have still to show that (ii) implies (iii). Hence assume that U has the SSW_n-property and suppose $U \notin WT_n$. Then there exists some $u_0 \in U \backslash \{0\}$ with at least n sign changes in (a, b). Since U_0 is regular, this means that there exist $a < z_0 < t_1 < z_1 < \cdots < z_{n-1} < t_n < z_n < b$ such that $u_0(z_i)\, u_0(z_{i+1}) < 0$, $i = 0, ..., n-1$, $u_0(t_i) = 0$ and $t_i \notin Z(U)$, $i = 1, ..., n$.

Let $T := \{t_1, ..., t_n\}$. Since $T \subset (a, b) \backslash Z(U)$, as in the proof of Theorem 4.6 we may assume that no knot interval K_i is contained in $Z(U)$.

Moreover, we then can choose u_0 such that u_0 does not vanish identically in K_i, $i = 0, ..., r$. To show this suppose that $u_0 \equiv 0$ in K_{i_0} for some $i_0 \in \{0, ..., r\}$. Since $K_{i_0} \not\subset Z(U)$, there exists $\tilde{u} \in U$ with $\tilde{u} \not\equiv 0$ in K_{i_0}. Then for some sufficiently small ε the function $\tilde{u}_0 := u_0 + \varepsilon \tilde{u}$ has at least n sign changes and n zeros $\{\tilde{t}_i\}_{i=1}^{n} \subset (a, b) \backslash Z(U)$. In addition, we have that $\tilde{u}_0 \not\equiv 0$ in K_{i_0} and if $\tilde{u}_0 \equiv 0$ in some K_i, $i \neq i_0$, then $u_0 \equiv 0$ there. Continuing this method for some zero interval of $\tilde{u}_0$, after a finite number of steps we obtain a function with at least n sign changes and n zeros in $(a, b) \backslash Z(U)$ which does not vanish identically in any K_i, $i = 0, ..., r$. Hence we may assume that u_0 has this property.

Let $T = \{t_1, ..., t_n\}$ being defined as above. Then T fails to be an SSW-set w.r.t. U. Otherwise by the SSW_n-property of U, T would be an I-set w.r.t. U which contradicts the fact that $T \subset Z(u_0)$ and $u_0 \not\equiv 0$.

Since $\text{card}(T) = n = \dim U$, it then follows that there exists a subinterval $I := [x_i, x_j]$ of $[a, b]$ such that

$$\text{card}(T \cap I) \geqslant n_{ij} := \dim U|_I$$

and

$$\text{card}(T \cap \tilde{I}) \leqslant \dim U|_{\tilde{I}}$$

for every proper subinterval $\tilde{I}$ of I. We consider two cases.

Case 1. Suppose that

$$\text{card}(T \cap R) \leqslant \dim U|_R$$

for every $R := \bigcup_{k=1}^{l} K_{i_k} \subset I$.

Choose a subset $\tilde{T} \subset T \cap I$ such that $\text{card}(\tilde{T}) = n_{ij}$. In particular, it follows that

$$\text{card}(\tilde{T} \cap R) \leqslant \dim U|_R$$

for every R as above. (Note that in this case I must be a proper subset of $[a, b]$, since T fails to be an SSW-set w.r.t. U.)

Case 2. Suppose that

$$\text{card}(T \cap R) > \dim U|_R$$

for some $R := \bigcup_{k=1}^{l} K_{i_k} \subset I$ and assume that the number l of knot intervals of R is minimal in the sense that

$$\text{card}(T \cap \tilde{R}) \leqslant \dim U|_{\tilde{R}}$$

for every $\tilde{R} := \bigcup_{k=1}^{\tilde{l}} K_{m_k} \subset I$ where $\tilde{l} < l$.

Choose a subset $\tilde{T} \subset T \cap R$ such that $\text{card}(\tilde{T}) = \dim U|_R$. Then by assumption on R,

$$\text{card}(\tilde{T} \cap \tilde{R}) \leqslant \dim U|_{\tilde{R}}$$

for every $\tilde{R} := \bigcup_{k=1}^{\tilde{l}} K_{m_k} \subset R$.

Thus in the Cases 1 and 2 we have defined SSW-sets $\tilde{T}$ w.r.t. $U|_I$ and w.r.t. $U|_R$, respectively. Since Case 1 is obviously a special case of Case 2, we shall only consider this case and shall complete $\tilde{T}$ to obtain an SSW-set w.r.t. U.

Hence suppose that Case 2 is given. Recall that u_0 does not vanish identically in K_{i_k}, $k = 0, ..., l$. This implies that $u_0 \not\equiv 0$ in R. To simplify the following arguments we may assume that $x_i := x_{i_1} = \min R$, $x_j := x_{i_l+1} = \max R$ and define

$$\tilde{U} := \{u \in U \colon u \equiv 0 \text{ in } R\}.$$

Hence we have that $K_i = [x_i, x_{i+1}] \subset R$, $K_{j-1} = [x_{j-1}, x_j] \subset R$ and

$$n_{ij} := \dim U|_{[x_i, x_j]} = \dim U|_R + \dim \tilde{U}|_{[x_i, x_j]}.$$

We first complete $\tilde{T}$ to obtain an SSW-set w.r.t. $U|_{[x_i, x_j]}$. We define

$$U_{i+1} := \tilde{U},$$

$$U_{q+1} := \{u \in U_q \colon u \equiv 0 \text{ in } K_q\}, \qquad q = i+1, ..., j-3.$$

Let $l_{q+1} := \dim U_{q+1}|_{K_{q+1}}$, $q = i, ..., j-3$. Then it is easily seen that $l_{q+1} = 0$ if $K_{q+1} \subset R$, $U_{j-2} \subset U_{j-3} \subset \cdots \subset U_{i+1}$ and

$$\sum_{q=i}^{j-3} l_{q+1} = \dim \tilde{U}|_{[x_i, x_j]}.$$

The last equality follows, because

$$\begin{aligned}\dim \tilde{U}|_{[x_i, x_j]} &= \dim \tilde{U}|_{[x_{i+1}, x_{j-1}]}\\ &= \dim \tilde{U}|_{K_{i+1}} + \dim\{u \in \tilde{U}: u \equiv 0 \text{ in } K_{i+1}\}|_{[x_{i+2}, x_{j-1}]}\\ &= l_{i+1} + \dim U_{i+2}|_{[x_{i+2}, x_{j-1}]}\\ &= l_{i+1} + l_{i+2} + \dim U_{i+3}|_{[x_{i+3}, x_{j-1}]} = \cdots = \sum_{q=i}^{j-3} l_{q+1}.\end{aligned}$$

We now want to show that there exists $u_{i+1} \in U_{i+1}$ such that the function $u_0 - u_{i+1}$ has at least l_{i+1} zeros in (x_{i+1}, x_{i+2}).

To prove it we first assume that $u_0 = \tilde{u}$ in K_{i+1} for some $\tilde{u} \in U_{i+1}$. Then we set $u_{i+1} := \tilde{u}$. Otherwise, the subspace $\operatorname{span}(\{u_0\} \cup U_{i+1})|_{K_{i+1}}$ has dimension $l_{i+1} + 1$ which implies that it must contain a function $u_0 - u_{i+1}$ with at least l_{i+1} zeros in (x_{i+1}, x_{i+2}).

In both cases we choose a subset T_{i+1} of (x_{i+1}, x_{i+2}) such that $T_{i+1} \subset Z(u_0 - u_{i+1})$ and $\operatorname{card}(T_{i+1}) = l_{i+1}$. (In the particular case when $K_{i+1} \subset R$, it follows that $U_{i+1}|_{K_{i+1}} = \{0\}$. Then $l_{i+1} = 0$ which implies that $T_{i+1} = \emptyset$.) Since $K_i \subset R$ and $U_{i+1} = \tilde{U}$, we have that $u_0 - u_{i+1} = u_0$ in R and $u_{i+1} \equiv 0$ in K_i. Therefore, $u_0 - u_{i+1} \not\equiv 0$ in R.

Continuing this method in K_{i+2} we find a function $u_{i+2} \in U_{i+2}$ such that $u_0 - u_{i+1} - u_{i+2}$ has at least l_{i+2} zeros in (x_{i+2}, x_{i+3}). Let $T_{i+2} \subset (x_{i+2}, x_{i+3})$ satisfying $T_{i+2} \subset Z(u_0 - u_{i+1} - u_{i+2})$ and $\operatorname{card}(T_{i+2}) = l_{i+2}$. (In the particular case when $K_{i+2} \subset R$, it follows that $l_{i+2} = 0$ and $T_{i+2} = \emptyset$.)

Moreover, in view of the properties of U_{i+2}, we have that $u_0 - u_{i+1} - u_{i+2} = u_0 \not\equiv 0$ in R and $u_0 - u_{i+1} - u_{i+2} = u_0 - u_{i+1}$ in $R \cup K_{i+1}$. In particular, $T_{i+1} \subset Z(u_0 - u_{i+1} - u_{i+2})$.

Continuing this process in $K_{i+3}, ..., K_{j-2}$ we finally get a function

$$\tilde{u}_0 := u_0 - \sum_{q=i+1}^{j-2} u_q,$$

and subsets T_q of (x_q, x_{q+1}) such that

$$T_q \subset Z(\tilde{u}_0), \qquad \operatorname{card}(T_q) = l_q, \qquad q = i+1, ..., j-2.$$

In particular, $T_q = \emptyset$ if $K_q \subset R$.

Moreover, since $\tilde{u}_0 = u_0$ in R, it follows that $\tilde{u}_0 \not\equiv 0$ in R, $\tilde{T} \subset Z(\tilde{u}_0)$ (where $\tilde{T}$ denotes the subset of $Z(u_0) \cap R$ which was defined in Case 2) and

$$\operatorname{card}\left(\tilde{T} \cup \bigcup_{q=i+1}^{j-2} T_q\right) = \dim U|_R + \sum_{q=i+1}^{j-2} l_q$$
$$= \dim U|_R + \dim \tilde{U}|_{[x_i, x_j]} = n_{ij}.$$

We set

$$\hat{T} := \tilde{T} \cup \bigcup_{q=i+1}^{j-2} T_q.$$

Then by the above arguments, $\hat{T} \subset Z(\tilde{u}_0) \cap [x_i, x_j]$ and $\operatorname{card}(\hat{T}) = n_{ij} = \dim U|_{[x_i, x_j]}$.

We shall now show that $\hat{T}$ is an SSW-set w.r.t. $U|_{[x_i, x_j]}$. Assuming that

$$\tilde{R} := \bigcup_{k=1}^{\bar{l}} K_{m_k} \subset [x_i, x_j]$$

where $i \leqslant m_1 < \cdots < m_{\bar{l}} \leqslant j-1$ we have to show that

$$\operatorname{card}(\hat{T} \cap \tilde{R}) \leqslant \dim U|_{\tilde{R}}. \tag{6.2}$$

We consider three cases.

(i) Suppose that $\tilde{R} \subset R$. Then $\hat{T} \cap \tilde{R} = \tilde{T} \cap \tilde{R}$ and by the assumption on R in Case 2, (6.2) follows immediately.

(ii) Suppose that $\operatorname{int}_K \tilde{R} \cap R = \varnothing$. From the choice of T_q, $q = i+1, ..., j-2$ it then follows that $\hat{T} \cap \tilde{R} = \bigcup_{k=1}^{\bar{l}} T_{m_k}$. This implies that

$$\operatorname{card}(\hat{T} \cap \tilde{R}) = \sum_{k=1}^{\bar{l}} l_{m_k} = \sum_{k=1}^{\bar{l}} \dim U_{m_k}|_{K_{m_k}}$$
$$\leqslant \dim \tilde{U}|_{\tilde{R}} \leqslant \dim U|_{\tilde{R}}$$

where the first inequality follows from the fact that $U_{m_{\bar{l}}} \subset U_{m_{\bar{l}-1}} \subset \cdots \subset U_{m_1} \subset \tilde{U}$ and

$$\sum_{k=1}^{\bar{l}} l_{m_k} \leqslant \dim U_{m_1}|_{\tilde{R}} \leqslant \dim \tilde{U}|_{\tilde{R}}.$$

(iii) Suppose that $\tilde{R} = \tilde{R}_1 \cup \tilde{R}_2$ where $\tilde{R}_1 \subset R$ and $\operatorname{int}_K \tilde{R}_2 \cap R = \varnothing$. Concluding as in (i) and (ii) we then have that

$$\operatorname{card}(\hat{T} \cap \tilde{R}_1) \leqslant \dim U|_{\tilde{R}_1}$$

and

$$\text{card}(\hat{T} \cap \tilde{R}_2) \leqslant \dim \tilde{U}|_{\tilde{R}_2}.$$

Let $\hat{U} := \{u \in U : u \equiv 0 \text{ in } \tilde{R}_1\}$. Since $\tilde{R}_1 \subset R$, it then follows that $\tilde{U} \subset \hat{U}$. Moreover, we have that

$$\dim U|_{\tilde{R}} = \dim U|_{\tilde{R}_1} + \dim \hat{U}|_{\tilde{R}_2}.$$

Using these arguments we finally obtain that

$$\begin{aligned} \text{card}(\hat{T} \cap \tilde{R}) &= \text{card}(\hat{T} \cap \tilde{R}_1) + \text{card}(\hat{T} \cap \tilde{R}_2) \\ &\leqslant \dim U|_{\tilde{R}_1} + \dim \tilde{U}|_{\tilde{R}_2} \\ &\leqslant \dim U|_{\tilde{R}_1} + \dim \hat{U}|_{\tilde{R}_2} = \dim U|_{\tilde{R}}. \end{aligned}$$

This proves (6.2). ∎

Thus we have shown that $\hat{T}$ is an SSW-set w.r.t. $U|_{[x_i, x_j]}$ where $\hat{T} \subset Z(\tilde{u}_0)$ and $\tilde{u}_0 \not\equiv 0$ in $[x_i, x_j]$.

We now define

$$\begin{aligned} U_j &:= \{u \in U : u \equiv 0 \text{ in } [x_i, x_j]\}, \\ U_{q+1} &:= \{u \in U_q : u \equiv 0 \text{ in } K_q\}, \qquad q = j, ..., r-1 \end{aligned}$$

and

$$l_q := \dim U_q|_{K_q}, \qquad q = j, ..., r.$$

Analogously as above we complete $\hat{T}$ to a subset $\bar{T}$ of $[x_i, x_{r+1}]$ such that $\bar{T} \subset Z(\bar{u}_0)$ for some $\bar{u}_0 \in U$, $\bar{u}_0 \not\equiv 0$ in $[x_i, x_{r+1}]$, $\bar{T} \cap [x_i, x_j] = \hat{T}$, $\text{card}(\bar{T} \cap (x_j, x_{r+1}]) = \sum_{q=j}^{r} l_q = \dim U_j|_{[x_j, x_{r+1}]}$ which implies that $\text{card}(\bar{T}) = n_{ij} + \dim U_j|_{[x_j, x_{r+1}]} = \dim U|_{[x_i, x_{r+1}]}$, and $\bar{T}$ is an SSW-set w.r.t. $U|_{[x_i, x_{r+1}]}$.

We finally apply the above method to the interval $[x_0, x_i]$ and the function $\bar{u}_0$ and obtain a function $\hat{u} \in U \backslash \{0\}$ and a subset $T(\hat{u}) \subset Z(\hat{u})$ such that $T(\hat{u})$ is an SSW-set w.r.t. U. But this contradicts the hypothesis on U to have the SSW_n-property.

Thus we have shown that $U \in WT_n$ and the proof of Theorem 4.9 is completed. ∎

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